A problem with non-local conditions for a mixed parabolic-hyperbolic equation with two lines of changing type

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I. Introduction. Formulation of the problems

The theory of mixed type equations is one of the modern part of the theory of partial differential equations. Recently a circle of problems for mixed type equations was considerably extended. Studying boundary-value problems for mixed parabolic-hyperbolic type equations is also one of the actual directions of the theory of mixed type equations. It can be explained on the one hand mathematical models of some real-life processes are brought to study problems for such type equations, on the other hand it is inner neseccity of the theory of the theory of partial differential equations. For instance, for the first time the necessity of consideration of parabolic-hyperbolic type equation was specified in 1959 by I. M. Gel'fand [1]. He gave an example, connected to the movement of gas in a channel, surrounded by a porous environment: inside the channel the movement of the gas is described by the wave equation, outside by the equation of diffusion. The basic bibliography about the history of the occurrence and development of this subject can be found in the book of T. D. Djuraev [2]. At the present time researchers pay attention to study problems with nonlocal conditions, such as problems with integral conditions, with Bitsadze-Samarskiy conditions and others. For more information we note works [3],[4] and references therein.

Consider the following parabolic-hyperbolic equation

$$0 = L_\lambda u \equiv \begin{cases} u_{xx} - u_y - \lambda_1^2 u, & (x,y)\in\Omega_0, \\ u_{xx}\,sign\, y + u_{yy}\,sign\, x - \lambda_2^2 u\, sign(x+y), & (x,y)\in\Omega_1\cup\Omega_2\cup\Omega_1^*\cup\Omega_2^*, \end{cases} \quad (1)$$

where λ_1, λ_2 are given complex numbers, $\Omega_0 = \{(x,y): 0<x<1, 0<y<1\}$, $\Omega_1 = \{(x,y): -y<x<1+y, (-1/2)<y<0\}$, $\Omega_2 = \{(x,y): -x<y<1+x, (-1/2)<x<0\}$, and also Ω_1^* and Ω_2^* are domains symmetric domains Ω_1 and Ω_2 with respect to line $x+y=0$ respectively.

Let $O(0,0)$, $A(1,0)$, $B(0,1)$, $A^*(0,-1)$, $B^*(-1,0)$, $C(1/2,-1/2)$, $D(-1/2,1/2)$, $A_0(1,1)$, a $OA(OB^*)$, $OB(OA^*)$, $OC(OD)$, AA_0 is the segment of the lines $y=0$, $x=0$, $x+y=0$, $x=1$ respectively; $\Omega = \Omega_0\cup\Omega_1\cup\Omega_2\cup\Omega_1^*\cup\Omega_2^*\cup OA\cup OB\cup OC\cup OD$.

Boundary value problems for equation $L_\lambda u = 0$ in the domain $\Omega_0\cup\Omega_1\cup OA$ investigated in the works [2,5,6], and in the domain $\Omega_0\cup\Omega_1\cup\Omega_2\cup OA\cup OB$ in [7]. And also various type

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of boundary value problems were formulated and investigated in [8,11,12,13] for equation $L_\lambda u = 0$ in the domain Ω.

In this paper we study the following problem for equation (1) in the Ω.

Problem M. Find a function $u(x,y)$ satisfying following condition:

1) It is a regular solution of the equation $L_\lambda u = 0$ in the domains Ω_0, Ω_1, Ω_2, Ω_1^*, Ω_2^*;
2) $u(x,y) \in C(\overline{\Omega}) \cap C^1((\Omega \cup OA^* \cup OB^*) \setminus (OC \cup OD))$;
3) It satisfies conditions

$$u(1,y) = \varphi(y), \quad 0 \le y \le 1; \tag{2}$$

$$u_y(t,0) = f_1(t), \quad -1 < t < 0; \tag{3}$$

$$u_x(0,t) = f_2(t), \quad -1 < t < 0; \tag{4}$$

$$u(t,0) + u(0,-t) = g_1(t), \quad 0 \le t \le 1; \tag{5}$$

$$u(0,t) + u(-t,0) = g_2(t), \quad 0 \le t \le 1. \tag{6}$$

Here $\varphi(y)$, $f_j(t)$, $g_j(t)$ $(j=1,2)$ are given functions, such that $\varphi(y) \in C^1[0,1]$; $g_1(t)$, $g_2(t) \in C[-1,0] \cap C^2(-1,0)$ и $g_1(0) = g_2(0) = 0$; $f_1(t), f_2(t) \in C^1(-1,0)$ and may has singularity less that one when $t \to 0$ and $t \to (-1)$.

Let $u(x,y)$ be a solution of the problem M. We introduce notations:

$$u(x,0) = \tau_1(x), \quad 0 \le x \le 1; \quad u(0,y) = \tau_2(y), \quad 0 \le y \le 1;$$

$$u_y(x,0) = \nu_1(x), \quad 0 < x < 1; \quad u_x(0,y) = \nu_2(y), \quad 0 < y < 1.$$

Then using conditions (3)-(6) and formulas which define solution of the problem Cauchy for equation $L_\lambda u = 0$ in the domains Ω_j, Ω_j^* $(j=1,2)$ [9], and also continuity of the solution $u(x,y)$ for transition over the line $x + y = 0$, it is easy to see that, the problem M is equivalent to the following problem in Ω_0 : find a regular in the domain Ω_0 solution $u(x,y) \in C(\overline{\Omega}_0) \cap C^1(\Omega_0 \cup OA \cup OB)$ of the equation

$$u_{xx} - u_y - \lambda_1^2 u = 0, \quad (x,y) \in \Omega_0, \tag{7}$$

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satisfying conditions (2) and

$$\tau_1(x)=\frac{1}{2}\int_0^x\left[\nu_1(t)+f_2(-t)\right]J_0\left[\lambda_2(x-t)\right]dt+\frac{1}{2}g_1(x),\qquad 0\le x\le 1; \tag{8}$$

$$\tau_2(x)=\frac{1}{2}\int_0^x\left[\nu_2(t)+f_1(-t)\right]J_0\left[\lambda_2(x-t)\right]dt+\frac{1}{2}g_2(x),\qquad 0\le x\le 1; \tag{9}$$

where $J_0(z)$ is first kind zero index Bessel function.

Assuming (8) and (9), as Abels inetgral equation with respect to $\nu_1(x)+f_2(-x)$ and $\nu_2(x)+f_1(-x)$ respectively, as in [10], we obtain

$$\nu_1(x)=-f_2(-x)+C_{0x}^{0,\lambda_2}[2\tau_1(x)-g_1(x)],\qquad 0<x<1; \tag{10}$$

$$\nu_2(x)=-f_1(-x)+C_{0x}^{0,\lambda_2}[2\tau_2(x)-g_2(x)],\qquad 0<x<1, \tag{11}$$

where $C_{mx}^{0,\lambda}[p(x)]\equiv sign(x-m)\left\{\frac{d}{dx}p(x)+\frac{1}{2}\lambda^2\int_m^x p(t)\bar{J}_1[\lambda(x-t)]dt\right\},\bar{J}_1(z)=(2/z)J_1(z)$.

(10) and (11) are the main functional relations among $\tau_1(x)$, $\tau_2(x)$, $\nu_1(x)$ and $\nu_2(x)$, obtained from the condition thast the solution of problem M in the area of hyperbolicity of equation (1) must satisfy conditions (3)-(6).

II. The uniqueness of the solution.

Theorem. If the inequality $|l_1|$i $(\sqrt{2}/2)$ holds, then problem M cannot have more than one solution.

Proof. Let $u(x,y)$ be solution of the problem M for $j\ (y)$ є $f_j(y)$ є $g_j(x)$ є 0 $(j=\overline{1,2})$. Then the identity (7) and equalities are valid: $t_j(0)=t_j(1)=0$, $j=\overline{1,2}$,

$$n_1(x)=2C_{0x}^{0,l_2}[t_1(x)],\qquad n_2(x)=2C_{0x}^{0,l_2}[t_2(x)],\qquad 0<x<1. \tag{12}$$

Multiplying identity (7) by the function $u(x,y)$, rewrite in the form

$$(uu_x)_x-\frac{1}{2}(u^2)_y-(u_x)^2-l_1^2u^2=0,\ (x,y)\,\mathrm{O}W_0. \tag{13}$$

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Integrate identity (13) over the rectangle $\Omega_0^{\varepsilon,\eta}$, bounded by lines $x=\varepsilon$, $x=1-\varepsilon$, $y=\eta$, $y=1$, where ε and η are small enough positive numbers. Then, applying Green's formula, we have

$$\iint_{\Omega_0^{\varepsilon,\eta}} \left\{\lambda_1^2 u^2(x,y)+\left[u_x(x,y)\right]^2\right\}dxdy+\frac{1}{2}\int_{\varepsilon}^{1-\varepsilon} u^2(x,1)dx-\frac{1}{2}\int_{\varepsilon}^{1-\varepsilon} u^2(x,\eta)dx+$$

$$+\int_{\eta}^{1} u(\varepsilon,y)u_x(\varepsilon,y)dy-\int_{\eta}^{1} u(1-\varepsilon,y)u_x(1-\varepsilon,y)dy=0.$$

Hence, for $\eta\to 0$, $\varepsilon\to 0$, taking into account $u(1,y)\equiv 0$, we get

$$\iint_{\Omega_0} \left\{\lambda_1^2 u^2(x,y)+\left[u_x(x,y)\right]^2\right\}dxdy+\frac{1}{2}\int_0^1 u^2(x,1)dx-$$

$$-\frac{1}{2}\int_0^1 \tau_1^2(x)dx+\int_0^1 \tau_2(y)\nu_2(y)dy=0. \qquad (14)$$

Using equality (12), it is easy to verify that

$$\int_0^1 \tau_1(x)\nu_1(x)dx+\int_0^1 \tau_2(x)\nu_2(x)dx=\int_0^1 \left[\tau_1^2(x)+\tau_2^2(x)\right]'_x dx+$$

$$+\lambda_2^2\int_0^1 \tau_1(x)dx\int_0^x \tau_1(t)\bar{J}_1\left[\lambda_2(x-t)\right]dt+\lambda_2^2\int_0^1 \tau_2(x)dx\int_0^x \tau_2(t)\bar{J}_1\left[\lambda_2(x-t)\right]dt,$$

hence, by virtue of $\tau_j(0)=\tau_j(1)=0$, $j=\overline{1,2}$, we get

$$\int_0^1 \tau_1(x)\nu_1(x)dx+\int_0^1 \tau_2(x)\nu_2(x)dx=$$

$$=\lambda_2^2\int_0^1 \tau_1(x)dx\int_0^x \tau_1(t)\bar{J}_1\left[\lambda_2(x-t)\right]dt+\lambda_2^2\int_0^1 \tau_2(x)dx\int_0^x \tau_2(t)\bar{J}_1\left[\lambda_2(x-t)\right]dt. \qquad (15)$$

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Further, we integrate identity (13) with respect to the segment $\{(x,y): y=\eta,\ \varepsilon<x<1-\varepsilon\}$. In the resulting equality, passing to the limit at $\eta\to 0$, $\varepsilon\to 0$, taking into account $\tau_1(0)=0$ and $\tau_1(1)=0$, we get

$$\int_0^1 \tau_1(x)\nu_1(x)dx=-\int_0^1\left[\tau_1'(x)\right]^2dx-\lambda_1^2\int_0^1\tau_1^2(x)dx. \tag{16}$$

Substituting (16) into equality (15), we have

$$\int_0^1 \tau_2(x)\nu_2(x)dx=\frac{1}{2}\lambda_2^2\int_0^1\tau_1(x)dx\int_0^x\tau_1(t)\bar{J}_1\left[\lambda_2(x-t)\right]dt+$$

$$+\frac{1}{2}\lambda_2^2\int_0^1\tau_2(x)dx\int_0^x\tau_2(t)\bar{J}_1\left[\lambda_2(x-t)\right]dt+\int_0^1\left[\tau_1'(x)\right]^2dx+\lambda_1^2\int_0^1\tau_1^2(x)dx. \tag{17}$$

Using formula

$$J_\omega(z)=\frac{(z/2)^\omega}{\sqrt{\pi}\,\Gamma(\omega+1/2)}\int_{-1}^1(1-x^2)^{\omega-(1/2)}\cos(xz)\,dx,\ \operatorname{Re}\omega>-1/2,$$

where $\Gamma(z)$ is Euler gamma function, it is easy to see that

$$\int_0^1\tau_j(x)dx\int_0^x\tau_j(t)\bar{J}_1\left[\lambda_2(x-t)\right]dt=$$

$$=\frac{1}{\pi}\int_{-1}^1(1-x^2)^{1/2}\left[\left(\int_0^1\tau_j(t)\cos(\lambda_2xt)dt\right)^2+\left(\int_0^1\tau_j(t)\sin(\lambda_2xt)dt\right)^2\right]dx\geq 0,\ j=\overline{1,2}. \tag{18}$$

Substituting (17) into (14), and then taking into account the inequalities (18) and the condition $|\lambda_1|\geq(\sqrt{2}/2)$, we derive that $u_x(x,y)\equiv 0$, i.e. $u(x,y)=w(y)$, $(x,y)\in\bar{\Omega}_0$. Since $u(1,y)\equiv 0,\ 0\leq y\leq 1$, then $w(y)\equiv 0,\ 0\leq y\leq 1$. Consequently, $u(x,y)\equiv 0,\ (x,y)\in\bar{\Omega}_0$.

Since $\tau_1(x)\equiv\tau_2(y)\equiv 0$ for $\varphi_j(y)\equiv f_j(x)\equiv g_j(x)\equiv 0,\ j=\overline{1,2}$, from (12) follows that $\nu_1(x)\equiv\nu_2(y)\equiv 0$. Then, solutions of the Cauchy problem for equation (1) with homogeneous initial data, $u(x,y)\equiv 0$ in $\bar{\Omega}_1\cup\bar{\Omega}_3$. Consequently, $u(x,y)|_{\overline{ED}}\equiv 0$.

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Considering this and $f_1(x) \equiv f_2(y) \equiv 0$, then, according to the uniqueness of the solution of the Goursat problem for equation (1) in the domains Ω_2 and Ω_4, we get $u(x,y) \equiv 0$, $(x,y) \in \bar{\Omega}_2 \cup \bar{\Omega}_4$. Consequently, $u(x,y) \equiv 0$, $(x,y) \in \bar{\Omega}$, hence follows that Problem M cannot have more than one solution. The theorem has been proven.

III. Existence of the solution. Let $u(x,y)$ be a solution of the problem M. Then equality (7), (10) and (11) are valid. Passing to the limit at $y \to +0$, from (7) we obtain equation $\tau_1''(x) - \lambda_1^2 \tau_1(x) = \nu_1(x)$, $0 \le x \le 1$.

The solution of this equation in the interval $0 \le x \le 1$, satisfying the boundary conditions $\tau_1(0) = g_1(0)$, $\tau_1(1) = \varphi(0)$, is represented as

$$\tau_1(x) = F_1(x) + \int_0^1 K(x,t)\nu_1(t)\,dt, \qquad (19)$$

where $F_1(x) = x\varphi(0) + (1-x)g_1(0) + \lambda_1^2 \int_0^1 G(x,t)\left[t\varphi(0) + (1-t)g_1(0)\right]dt$,

$$K(x,t) = \begin{cases} sh\lambda_1(1-t)\,sh\lambda_1 x/(\lambda_1 sh\lambda_1), & 0 \le x \le t, \\ sh\lambda_1 t\ sh\lambda_1(1-x)/(\lambda_1 sh\lambda_1), & t \le x \le 1. \end{cases}$$

Substituting (10) into (19), we obtain

$$\tau_1(x) - \int_0^1 K_1(x,t)\tau_1(t)\,dt = F_2(x), \qquad 0 \le x \le 1. \qquad (20)$$

Here $G_1(x,t) = 2C_{1t}^{0,\lambda_2}[G(x,t)]$,

$$F_2(x) = F_1(x) - \int_0^1 f_2(-t)K(x,t)\,dt - \int_0^1 g_1(t)C_{1t}^{0,\lambda_2}\left[K(x,t)\right]dt.$$

(20) is an integral Fredholm equation of the second kind with respect to $\tau_1(x)$. The unique and unconditional solvability of Eq. (20), by virtue of equivalence, follows from the uniqueness of the solution to Problem M.

Next, in the domain Ω_0 we consider the first boundary value problem for Eq. (7), with boundary data $u(0,y) = \tau_2(y)$, $u(x,0) = \tau_1(x)$, $u(1,y) = \varphi(y)$.

The solution to this problem is determined by the formula [1]

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$$u(x,y)=\int_0^y \tau_2(\eta)e^{\lambda_1^2(\eta-y)}G_\xi(x,y;0,\eta)d\eta-$$

$$-\int_0^y \varphi(\eta)e^{\lambda_1^2(\eta-y)}G_\xi(x,y;1,\eta)d\eta+\int_0^1 \tau_1(\xi)e^{-\lambda_1^2 y}G(x,y;\xi,0)d\xi, \qquad (21)$$

where $G(x,y;\xi,\eta)=\frac{1}{2\sqrt{\pi(y-\eta)}}\sum_{n=-\infty}^{+\infty}\left\{\exp\left[-\frac{(x-\xi+2n)^2}{4(y-\eta)}\right]-\exp\left[-\frac{(x+\xi+2n)^2}{4(y-\eta)}\right]\right\}.$

Differentiating (21) with respect to x and setting $x=0$, after some calculations, we have

$$\nu_2(y)=-\int_0^y\left[\tau_2(\eta)e^{\lambda_1^2(\eta-y)}\right]'_\eta N(0,y;0,\eta)d\eta+F_3(y), \quad 0<y<1, \qquad (22)$$

where $F_3(y)=\int_0^y\left[\varphi(\eta)e^{\lambda_1^2(\eta-y)}\right]'_\eta N(0,y;1,\eta)d\eta+\int_0^1\tau_1'(\xi)e^{-\lambda_1^2 y}N(0,y;\xi,0)d\xi,$

$$N(0,y;\xi,\eta)=1\Big/\left[\pi(y-\eta)\right]^{-1/2}\sum_{n=-\infty}^{+\infty}\exp\left[-(\xi-2n)^2\Big/4(y-\eta)\right].$$

Substituting (9) into (22), we have

$$\nu_2(y)=\int_0^1 K_2(y,t)\nu_2(t)\,dt+F_4(y), \qquad 0<y<1, \qquad (23)$$

where

$$K_2(y,t)=\frac{1}{2}e^{\lambda_1^2(t-y)}N(0,y;0,t)+\frac{1}{2}\int_t^y\left[J_0[\lambda_2(\eta-t)]e^{\lambda_1^2(\eta-y)}\right]'_\eta N(0,y;0,\eta)d\eta,$$

$$F_4(y)=F_3(y)-\frac{1}{2}\int_0^y f_1(-\eta)e^{\lambda_1^2(\eta-y)}N(0,y;0,\eta)d\eta+$$

$$-\frac{1}{2}\int_0^y f_1(-t)dt\int_t^y\left[J_0[\lambda_2(\eta-t)]e^{\lambda_1^2(\eta-y)}\right]'_\eta N(0,y;0,\eta)d\eta-$$

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$$-\frac{1}{2}\int_0^y \left[g_2(h)e^{\lambda_1^2(h-y)} \right]_h N(0,y;0,h)dh.$$

(23) is a Volterra integral equation of the second kind with a weak singularity with respect to $n_2(y)$. It has unique solution.

After the function $\nu_2(x)$ from (23) is found, the functions $\tau_2(x)$ and $\nu_1(x)$ are uniquely found by formulas (9) and (10), respectively. After that, the solution of problem M in the domain Ω_0 is defined by (21), and in the domains Ω_j, Ω_j^* $(j=1,2)$ fomulas that gives the solution of the Cauchy problem for the equation $L_\lambda u=0$ [9].

This completes the proof of the existence of a solution to Problem M.

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