OPTIMAL QUADRATURE FORMULA FOR THE APPROXIMATION OF THE RIGHT RIEMANN-LIOUVILLE INTEGRAL

[1]Khaydarov I.U., [2]Turgunboyev B.Sh. [3]Qudratillayev M.

[1,2,3] Ferghana State University (Ferghana, Murabbiylar st., 19)

Abstract: In the present article, the problem of construction of the optimal quadrature formula in the sense of Sard is discussed for numerical integration of the right Riemann-Liouville integral in the Hilbert space of real-valued functions. Initially, the norm of the error functional is found using the extremal function of the error functional of the quadrature formula. Since the error functional is defined on the Hilbert space, the quadrature formula that we are constructing is exact for zeros of this space, that is, we have the conditions that the influence of the error functional on these functions is equal to zero. Then, the Lagrange function is constructed to find the conditional extremum of the error functional. Thereby, a system of linear equations is obtained for the coefficients of the optimal quadrature formula. The existence and uniqueness of the solution of the obtained system are studied.

Keywords: optimal quadrature formula, the extremal function, the error functional, optimal coefficient, Lagrange function.

1. Introduction.

It is known that calculus means integration and differentiation. Fractional calculus, as its name suggests, refers to fractional integration and fractional differentiation. Fractional integration often means Riemann-Liouville integral. But for fractional differentiation, there are several kinds of fractional derivatives. In the following definition is introduced [1, 3, 4].

Definition 1. (Definition 1 in [7]) The left fractional integral (or the left Riemann-Liouville integral) and right fractional integral (or the right Riemann-Liouville integral) with order $\alpha > 0$ of the given function $\varphi(t)$, $t \in (0,1)$ are defined a

$$D_{0,t}^{-\alpha}\varphi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1}\varphi(x)dx, \quad (1)$$

and

$$D_{t,1}^{-\alpha}\varphi(t) = \frac{1}{\Gamma(\alpha)}\int_t^1 (x-t)^{\alpha-1}\varphi(x)dx, \quad (2)$$

respectively, where $\Gamma(\alpha)$ is Eulier's gamma function.

The study of fractional integrals (1) and (2) is a two hundred year old subject that is part of a branch of mathematical analysis called Fractional Calculus [1, 2, 4]. Recently, due to its many applications in science and engineering, there has been an increase of interest in the study of fractional calculus. Fractional integrals appear naturally in many different contexts, e.g., when dealing with fractional variational problems or fractional optimal control. And also at the present time researchers pay attention to study partial differential equations including fractional differential operators and problems related to these equations. For more information we note works [8], [9] and references therein. While studying problems for mixed type equations one has to solve some integral equations including fractional differential equations (see [10], [11], [12],[13]). As is frequently observed, solving such equations analytically can be a difficult task, even impossible in some cases. One way to overcome the problem consists of applying numerical methods, e.g., using Riemann sums to approximate the fractional operators.

ISSN 2277-3630 (online), Published by International journal of Social Sciences & Interdisciplinary Research., under Volume: 11 Issue: 12 in December-2022
https://www.gejournal.net/index.php/IJSSIR

In this work we construct a quadrature formula for approximation of the right Riemann–Liouville integral (2). We consider a quadrature formula of the following form

$$\int_t^1 \frac{\varphi(x)dx}{\sqrt{x-t}} \cong \sum_{\beta=0}^{N} C_\beta \varphi(x_\beta) \tag{3}$$

where $h=\frac{1-t}{N},\ t\le 1,\ x_\beta = h\beta + t$, and function φ belongs to the linear space $\varphi(x)\in L_2^{(1)}(t,1)$, which is defined as (see [1, 2]).

The following difference is called the error of the quadrature formula (3)

$$(l,\varphi)=\int_t^1 \frac{\varphi(x)dx}{\sqrt{x-t}} - \sum_{\beta=0}^{N} C_\beta \varphi(x_\beta) = \int_{-\infty}^{\infty} l(x)\varphi(x)dx$$

The error functional for quadrature formula (3) has the following form

$$\ell(x)=\frac{\varepsilon_{[t,1]}(x)}{\sqrt{x-t}} - \sum_{\beta=0}^{N} C_\beta \delta(x-x_\beta)$$

Here, are coefficients of quadrature formula (3), $\varepsilon_{[t,1]}(x)$ is the characteristic function of the interval $[t,1]$, and $\delta(x)$ is Dirac's delta-function.

The inner product of two functions $\varphi(x)$ and $\psi(x)$ in the space $L_2^{(1)}(t,1)$ is defined as

$$\langle \varphi,\psi\rangle = \int_t^1 \left(\varphi^{(1)}\right)\left(\psi^{(1)}\right)dx$$

And norm of the function in this space is determined as

$$\|\varphi\|_{L_2^{(1)}} = \sqrt{\int_t^1 \left(\varphi^{(1)}(x)\right)^2 dx}.$$

The error of the quadrature formula is a linear functional in $L_2^{(1)}(t,1)$, where $L_2^{(1)*}(t,1)$ is the conjugate space to the space $L_2^{(1)}(t,1)$.

It is natural to evaluate the quality of the quadrature formula (3) using the maximum error of this formula on the unit ball of the Hilbert space $L_2^{(1)}(t,1)$, that is, using the norm of the functional

$$\|\ell\|_{L_2^{(1)*}} = \sup_{\|\varphi\|_{L_2^{(1)*}}=1} |(\ell,\varphi)|$$

Obviously, the norm of the error functional depends on C_β coefficients and x_β nodes.

If

$$\left\|\mathring{\ell}\right\|_{L_2^{(1)*}} = \sup_{C_\beta, x_\beta} \|\ell\|_{L_2^{(1)*}}$$

then the functional $\mathring{\ell}$ is said to correspond to **the optimal quadrature formula** in $L_2^{(1)}(t,1)$.

Problem above in such a general formulation is quite difficult. Minimizing the norm of the error functional with respect to the coefficients C_β is a linear problem, and with respect to the nodes

ISSN 2277-3630 (online), Published by International journal of Social Sciences & Interdisciplinary Research., under Volume: 11 Issue: 12 in December-2022
https://www.gejournal.net/index.php/IJSSIR

x_β it is actually non-linear, complicated problem, therefore, for simplicity, we consider this problem with fixed nodes x_β.

The main problem, in this work, is as follows.

Problem 1. Find the coefficients C_β that give minimum value to, and calculate

$$\left\|\overset{\circ}{\ell}\right\|_{L_2^{(1)*}}=\inf_{C_\beta}\left\|\ell\right\|_{L_2^{(1)*}}.$$

For solving this problem, firstly, we must find the norm of the error functional. For this we need an extremal function of the error functional ℓ. General form of the extremal function in $L_2^{(1)}(t,1)$ space was found in works [2, 4, 5] and [10]. In particular, we get the following

$$\left\|\ell\right\|^2_{L_2^{(1)*}}=(\ell,\psi_\ell)=\int_{-\infty}^{\infty}\ell(x)\psi_\ell(x)dx \tag{4}$$

here ψ_ℓ is the extremal function and it is defined as follows

$$\psi_\ell=\ell(x)*G_1(x)+P_0(x) \tag{5}$$

herein

$$G_1(x)=\frac{|x|}{2}$$

Using (5) from relation (4) we get the following

$$\left\|\ell\right\|^2_{L_2^{(1)*}}=(\ell,\psi_\ell)=\int_{-\infty}^{\infty}\ell(x)\left[\ell(x)*G_1(x)+P_0(x)\right]dx \tag{6}$$

It should be noted that since the error functional $\ell(x)$ is defined on the space $L_2^{(1)}(t,1)$, it satisfies the following conditions

$$(\ell,1)=0. \tag{7}$$

The equalities (7) mean that the quadrature formula (3) is exact for any polynomial of degree $m=1$.

Then, from equality (6), taking into account the expression (7), we have the following

$$\left\|\ell\right\|^2_{L_2^{(1)*}}=\int_{-\infty}^{\infty}\ell(x)\left(\ell(x)*G_1(x)\right)dx=$$

$$=\int_{-\infty}^{\infty}\left(\frac{\varepsilon_{[t,1]}(x)}{\sqrt{x-t}}-\sum_{\beta=0}^{N}C_\beta\delta(x-x_\beta)\right)\left(\int_t^1\frac{G_1(x-x_\beta)}{\sqrt{y-t}}dy-\sum_{\gamma=0}^{N}C_\gamma G_1(x-x_\gamma)\right)dx.$$

From here we get

$$\left\|\ell\right\|^2_{L_2^{(1)*}}=(\ell,\psi_\ell)=\left[\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N}C_\beta C_\gamma G_1(h\beta-h\gamma)-\right.$$

$$\left.-2\sum_{\beta=0}^{N}C_\beta\int_t^1\frac{G_1(x-(h\beta+t))}{\sqrt{x-t}}dx+\int_t^1\int_t^1\frac{G_1(x-y)}{\sqrt{x-t}\sqrt{y-t}}dxdy\right] \tag{8}$$

ISSN 2277-3630 (online), Published by International journal of Social Sciences & Interdisciplinary Research., under Volume: 11 Issue: 12 in December-2022
https://www.gejournal.net/index.php/IJSSIR

The expression (8) is a multivariate function with respect to coefficients C_β. Given the condition (7), we consider the Lagrange function to find the minimum of the expression

$$\Phi(C,\lambda)=\|\ell\|^2-(\ell,1)+P_0(x)$$

here C, and λ.

Also, we have

$$(\ell,x^k)=\int_t^1\frac{dx}{(x-t)^{1-\alpha}}-\sum_{\beta=0}^{N}C_\beta x_\beta^k=0,\ k=0,1,..,m-1.$$

In that case, equating to **0** the partial derivatives of the function Φ by C and λ, we get the following system of linear equations

$$\sum_{\gamma=0}^{N}C_\gamma G_1(h\beta-h\gamma)+\lambda_0 x_\beta^0=f_1[\beta],\ \beta=0,1,...,N, \quad (9)$$

$$\sum_{\gamma=0}^{N}C_\gamma x_\gamma^0=g_0 \quad (10)$$

Here,

$$f_1[\beta]=\int_t^1\frac{G_m[x-(h\beta+t)]}{\sqrt{x-t}}dx, \quad (11)$$

$$g_0=\int_t^1\frac{dx}{\sqrt{x-t}}. \quad (12)$$

Thus, we calculate the above integrals and we get the following expressions

$$f_1[\beta]=\frac{4}{3}\sqrt{(h\beta)^3}+\frac{1}{3}\sqrt{(1-t)^3}+h\beta\sqrt{1-t},$$

Likewise,

$$g_0=2\sqrt{1-t}.$$

In this way, we have redefined the system of linear equations (11)-(12).

References

1. Kilbas A.A., Srivastava H.M., and Trujillo J.J. 2006. Theory and Applications of Fractional Differential Equations. // in: North–Holland Mathematics Studies, Elsevier, Amersterdam. – vol. 204..

2. Miller K.S. and Ross B. 1993. An Introduction to the Fractional Calculus and Fractional Differential Equations. // A Wiley–Interscience Publication Wiley, New York.

3. Podlubny I. 1999. Fractional Differential Equations. // Academic Press, New York.

4. Samko S.G., Kilbas A.A., and Marichev O.I. 1993. Fractional Integrals and Derivatives. //Theory and Applications. Gordon and Breach Science Publishers, Switzer-land.

5. Sobolev S.L. 2006. The coefficients of optimal quadrature formulas. // Selected Works of S.L. Sobolev, Springer US, – P. 561–566.

6. Sobolev S.L. 1974. Introduction to the theory of cubature formulas. // (Russian), Nauka, Moscow. – 808 p.

ISSN 2277-3630 (online), Published by International journal of Social Sciences & Interdisciplinary Research., under Volume: 11 Issue: 12 in December-2022
https://www.gejournal.net/index.php/IJSSIR

7. Changpin Li, and Zeng Fanhai 2015. Numerical Methods for Fractional Calculus. CRC Press, Taylor and Francis Group.

8. A.O.Mamanazarov. Nonlocal problems for a fractional order mixed parabolic equation. Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences. Vol. 4, № 1. 2021.

9. Маманазаров А.О. Краевые задачи для смешанно-параболического уравнения дробного порядка в неограниченных областях. Бюллетень Института математики. Т.4, № 6, с. 37-48.

10. I.U. Khaydarov. A problem with non-local conditions for a mixed parabolic-hyperbolic equation with two lines of changing type. International Journal of Research in Commerce, IT, Engineering and Social Sciences. V. 16, Issue 10. 2022

11. A.K.Urinov, Q.S.Khalilov, A Nonlocal Problem for a Third Order Parabolic-Hyperbolic Equation with a Singular Coefficient, J. Sib. Fed. Univ. Math. & Phys., 2022. 15(4), 467-481. DOI: 10.17516/1997-1397-2022-15-4-467-481.

12. А.К. Уринов, И.У. Хайдаров. Об одной задаче с нелокальными условиями для параболо-гиперболического уравнения. Доклады Адыгской (Черкесской) Международной академии наук. Т.12, № 2, 2010. С.80-87.

13. Уринов А.К., Халилов К.С. Об одной нелокальной задаче для параболо-гиперболических уравнений, Доклады АМАН. – Нальчик. 2013. Т.15. №1. – С. 24-30.

ISSN 2277-3630 (online), Published by International journal of Social Sciences & Interdisciplinary Research., under Volume: 11 Issue: 12 in December-2022
https://www.gejournal.net/index.php/IJSSIR