METHODS OF SETTING AND SOLVING MIXED AND BOUNDARY PROBLEMS FOR HYPERBOLIC TYPE EQUATIONS

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Abstract: In this article, the methods of solving these mixed and boundary problems are described. The Sturum-Liouville problem is described and the solution of such problems by the Fourier method is shown. The mixed problem put into the hyperbolic type equation was solved by the Fourier method.

Key words: initial condition, mixed problem, boundary value problem, general solution, eigenvalue, eigenfunction, Sturum-Liouville theorem and problem, Fourier method, Fourier coefficients.

The solutions of the equations of mathematical physics depend on the setting of the problem representing the initial and boundary conditions. In this case, firstly, the solution of the given problem must exist and be unique, and secondly, this solution must be stable (a small change in the solution corresponds to a slight change in the conditions). If the solution of the given problem exists, is unique and stable, then this problem is called a correctly posed (correct) problem.[8:221]

In practice, there are many ways to solve the above boundary problems, for example, the method of characteristics, the method of separation of variables, the method of resources, and the methods of approximate calculation. Some methods give the analytical expression of the solution of the equation (the problem posed to it), the second one only shows the existence of the solution, and the third one gives the numerical value of this solution. Even sometimes it is necessary to use different methods to find the solution of different problems for one equation.[4:24]

One of the most widely used methods for solving boundary and mixed-type problems in the theory of partial differential equations is the separation of variables or Fourier method. The mixed problem is set for hyperbolic and parabolic type equations; G∈R^n – and the initial and boundary conditions are given.[2:37] The idea of the Fourier method is as follows: the desired function, which depends on several variables, is assigned to a separate variable is searched in the form of a product of related functions. We begin the description of the mentioned method by searching for the solution of the first initial-boundary value problem for the narrow vibration equation with both ends fixed. The rest of the boundary problems and mixed problems are solved in a similar way. The following

$$u_{tt} = a^2 u_{xx} \qquad (1)$$

homogeneous of Eq

$$u(0,t) = 0,\ u(l,t) = 0 \qquad (2)$$

marginal and

$$u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) \qquad (3)$$

we find a solution that satisfies the initial conditions. Since the equation (1) is linear and homogeneous, the sum of its eigensolutions is also a solution of the equation. We can see their sum with some coefficients as the desired solution. First, we consider the following auxiliary problem:

$$u_{tt} = a^2 u_{xx}$$

of the equation is not exactly equal to zero

$$u(x,t) = X(x)T(t) \qquad (4)$$

multiplicative, where X, T are univariable, and functions of x and t are homogeneous

$$u(0,t) = 0,\ u(l,t) = 0 \qquad (5)$$

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be asked to find a solution that satisfies the boundary conditions[1].
Solution: Substituting expression (4) into equation (1).

$$X''(x)T(t) = \frac{1}{a^2}X(x)T''(t) \quad (6)$$

we form the equality , and from the equality (6).

$$\frac{X''(x)}{X(x)} = \frac{1}{a^2}\frac{T''(t)}{T(t)} \quad (7)$$

we form the equation. In order for the function defined by formula (4) to be a solution of equation (1), equation (7) must be satisfied for all values of the independent variables 0 < x < l, t > 0. [5:351] The left part of this equation is only x o , and the right-hand side are functions of only the t variable. Successively, assigning one of them and changing the other, we are sure that the left and right parts of equality (7) are equal to some fixed number:

$$\frac{X''(x)}{X(x)} = \frac{1}{a^2}\frac{T''(t)}{T(t)} = -\lambda \quad (8)$$

In this case, we took l with a negative sign for convenience in further calculations. (8) to find the functions X(x) and T(t) from the equations

$$X''(x) + \lambda X(x) = 0, X(x) \neq 0,$$
$$T''(x) + a^2\lambda T(x) = 0, T(t) \neq 0 \quad (9)$$

we arrive at ordinary differential equations and also from the boundary conditions (6).

$$u(0,t) = X(0) \qquad T(t) = 0,$$
$$u(l,t) = X(l) \qquad T(t) = 0$$

originates. Instead, the above equations show that in order for the function u(x,t) defined by formula (5) to be equal to zero, the function X(x) must satisfy the conditions X(0)= X(l) = 0, vice versa shows that T(t)=0. Therefore, from the given boundary conditions, we have conditions X(0)= X(l)= 0. So, during the solution of the given boundary value problem, we came to the following problem, known as the Sturm-Liouville problem for the function X(x):

Definition of λ

$$X''(x) + \lambda X(x) = 0, \; X(0) = X(1) = 0 \quad (10)$$

The value for which a nontrivial solution of the problem exists is called the eigenvalue of this problem, and the corresponding nontrivial solution is called the eigenfunction corresponding to the eigenvalue l. The set of all eigenvalues is called the spectrum of the problem. [] In general, the problem of finding the eigenvalues of differential equations and their corresponding eigenfunctions is called the Sturm-Liouville problem. In order to find a solution to the above-mentioned problem, we consider each of the negative, zero, and positive cases of λ separately. In this process, we use the knowledge known to us from simple differential equations.
Case 1. Let us assume that λ < 0. In this case, we know from the course of ordinary differential equations that (1.15) is the general solution of the second-order ordinary differential equation

$$X(x) = C_1 e^{\sqrt{-\lambda x}} + C_2 e^{\sqrt{-\lambda x}} \quad (11)$$

It will be in the form. Here C_1and〖C〗_2 are real numbers that are optional. We choose C_1 and 〖C〗_2 so that the boundary conditions in (10) are valid.

$$X(0) = C_1 + C_2 = 0 \;\;,\;\; X(l) = C_1 e^{\sqrt{-\lambda l}} + C_2 e^{\sqrt{-\lambda l}}$$

From this, we determine the following.

$$C_1 = -C_2, \; C_1\left(e^{\sqrt{-\lambda l}} - e^{-\lambda l}\right) = 0$$

In this case, considering that l < 0 and l<0, it follows from the second equation that C_1=0 and from the first equation that C_1 = C_2=0. If we conclude from this, if l < 0, the problem (10) has

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only 0 solution, so in this case it follows that the Sturm-Liouville problem does not have an eigenvalue and an eigenfunction.

Case 2. Let us assume that λ = 0. In this case, the second-order ordinary differential equation in equation (10) is of the form 〖X 〗^" (x)=0, the general solution of the equation

$$X(x) = C_1 x + C_2 \qquad (12)$$

consists of (12). Here, C_1, C_2 are arbitrary real numbers. We choose them from the boundary conditions in equation (10) as follows:

$$X(0) = C_2, X(1) = C_1 + C_2 = 0$$

It follows that

$$C_1 = C_2 = 0.$$

To sum up, according to (12), even in the case of λ = 0, equation (10) has only a zero solution, and the Sturm-Liouville equation does not have an eigenvalue and an eigenfunction. Let's look at the next case.

Case 3. Let us assume that λ > 0. In this case, from the course of differential equations, we know that the second-order ordinary differential equation in equation (10) has two joint complex characteristic solutions, and its general solution is

$$X(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x \qquad (13)$$

will be in the form of We choose the constants C_1 in such a way that the given boundary conditions in the conditions of equation (10) are appropriate, that is:

$$X(0) = C_1 = 0 \ , C_2 \sin\sqrt{\lambda}l = 0$$

$$X(x) \neq 0$$

taking into account that, we make sure that C_2≠0.

In this case

$$\sin\sqrt{\lambda}l = 0$$

it turns out to be. The solution of this trigonometric equation is as follows.

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2, \ n\epsilon Z$$

Thus, the problem (11) is not exactly equal to zero, i.e. non-trivial only when $\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2, \ n\epsilon$Z

$$X_n(x) = C_n \sin\frac{n\pi}{l}x$$

it follows that there are solutions. [9:96] Here C_n are arbitrary real numbers. (10) for the considered Sturm-Liouville problem

$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2$>0, n=1,2,…….., numbers are eigenvalues and

$$X_n(x) = \sin\frac{n\pi}{l}x$$

This is a scalar product of eigenfunctions

$$\left(X_n(x), X_m(x)\right) = \int_0^l X_n(x)X_m(x)dx = \int_0^l \sin\frac{\pi n}{l}x \sin\frac{\pi m}{l}xdx$$

$$= \frac{1}{2}\left[\int_0^l \cos\frac{(n-m)\pi}{l}x - \cos\frac{(n+m)\pi}{l}x\,dx\right] = \begin{cases}\frac{1}{2}, n = m\\ 0, n \neq m\end{cases}$$

equal to According to the above, we proved the following theorem:

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T h e o r e m a . (10) The Sturm-Liouville problem has a non-trivial solution only when $\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2$, all eigenvalues are positive and eigenfunctions corresponding to different eigenvalues are mutually orthogonal.[10:332]

Now we look for the solution of the second equation of (9) corresponding to the found eigenvalues $\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)$

$T''(x) + a^2 \lambda T(x) = 0, T(t) \neq 0$

Since the differential equation is similar to the first differential equation of (9) ($X(x)$ is replaced $T(t)$va λ is replaced by $a^2\lambda$ its general solution is in this form :

$$(t) = A_n \cos\frac{n\pi}{l}at + B_n \sin\frac{n\pi}{l}at \qquad (14)$$

In this case, according to equations (5), (13) and (14), the particular solution of the narrow free oscillation equation (5) satisfying the homogeneous boundary conditions has the following form:

$$u_n(x,t) = X_n(x)T_n(t) = \left(A_n \cos\frac{n\pi}{l}at + B_n \sin\frac{n\pi}{l}at\right)\sin\frac{n\pi}{l}ax$$

Since the above equation (12) is a linear and homogeneous second-order partial differential equation, the sum of the singular solutions found also satisfies the equation (12) and the boundary conditions (13):

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi}{l}at + B_n \sin\frac{n\pi}{l}at\right)\sin\frac{n\pi}{l}ax \qquad (15)$$

In this case, we choose the coefficients A_n and B_n so that the initial conditions (5) are fulfilled: A_n va B_n

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{l}x = \varphi(x)\,,$$

$$u_t(x,0) = \frac{n\pi}{l}a\sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x = \varphi(x) \qquad (16)$$

In this case, to find coefficients $A_n$va B_n from the obtained results, we use the fact that any continuously differentiable function f(x) defined in the range [0,l] can be expanded by sines (cosines) into a trigonometric series called a Fourier series, or the

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x$$

let it be Here, the numbers b_n, n ∈ N f(x) are called the Fourier coefficients of the function f(x) and they are determined using the following equation.

$$b_n = \frac{2}{l}\int_0^l \varphi_n \sin\frac{n\pi}{l}dx$$

is determined using equality. [3:131] Knowing this, in order to find A_n and $B_n$from equations (16), we expand the continuously differentiable functions φ(x) and πs(x) into the Fourier series and write the Fourier coefficients as follows:

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$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n \sin\frac{n\pi}{l} x dx, \qquad \varphi_n = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi}{l} x dx \qquad (17)$$

$$\psi(x) = \sum_{n=1}^{\infty} \psi_n \sin\frac{n\pi}{l} x dx, \qquad \psi_n(x) \frac{2}{l}\int_0^l \psi(x)\sin\frac{n\pi}{l} x dx \qquad (18)$$

$$A_n = \varphi_n = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi}{l} x dx$$

$$B_n = \frac{l}{\pi n a}\psi_n = \frac{2}{\pi n a}\int_0^l \psi(x)\sin\frac{n\pi}{l} x dx \qquad (19)$$

$A_{\,n}$ va B_n By putting the above values of (19) found through the formulas into the equation (15), we find the solution of the 1st type boundary value problem for the homogeneous narrow vibration equation (2)-(3) written in formal form. [6.100] Because, in this form written (15) yZechim is an infinite term series, and it is clear to us that this series may be divergent or its sum may not be differentiable. In these cases, we cannot consider the function represented by the line (15) as a solution to the problem under consideration. For this purpose, if we show that the functional series (15) whose coefficients are determined by the formulas (19) and the series formed as a result of its double differentiation are smooth convergent under certain conditions, the function defined by the series (15) is really (2)- (4) will consist of the solution of the problem.
Problem 1: for the equation $u_{tt} = a^2 u_{xx}$ t $0 < x < l, t > 0$ on the half-way

$$u(0,t) = u(l,t) = 0 \quad u|_{t=0} = \sin 7x, \quad u_t|_{t=0} = 0, a = 1\ , l = \pi$$

find a solution to the problem.
Solution: if we substitute $u(x,t) = X(x)T(t)$ and take into account that a=1

$$X''(x) + \lambda X(x) = 0$$
$$T''(x) + \lambda T(x) = 0$$

we have differential equations of the form Here l is an unknown constant parameter. Considering that u u(0,t) =0 , X(x)≠0 va $T(t) \neq 0$

$$X(0) = 0\ , X(l) = 0 \text{ va } T_t = 0 \qquad (20)$$

satisfies the conditions. The general solution of the differential equation

$$X(x) = C_1\cos\sqrt{\lambda x} + C_2 \sin\sqrt{\lambda x} \qquad (20\ )$$

considering the

$$C_1 = 0\,, \quad C_2 \sin\sqrt{\lambda l} = 0$$
$$C_2 \neq 0$$

We think that Otherwise, X(x)≡0 would have remained

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$$C_2 \sin\sqrt{\lambda l} = 0 \quad \Rightarrow \quad \sin\sqrt{\lambda l} = 0$$

$$\sqrt{\lambda l} = \pi + \pi n \quad \Rightarrow \quad \lambda = \left(\frac{\pi+\pi n}{l}\right)^2$$

From that $X_n(x) = sin\frac{\pi+\pi n}{l}x$, there is n=0,1,2,3,……..
it follows that According to the above results,

$$T_n(t) = a_n \cos\frac{\pi(1+n)}{l}t + b_n \sin\frac{\pi(1+n)}{l}t$$

$$u_n(x,t) = X_n(x)T_n(t) = sin\frac{\pi+\pi n}{l}x\left[a_n\, cos\frac{\pi(1+n)}{l}t + b_n\, sin\frac{\pi(1+n)}{l}t\right]$$

$$u(x,t) = \sum_{n=0}^{\infty} \sin\frac{\pi+\pi n}{l}x\left[a_n \cos\frac{\pi(1+n)}{l}t + b_n \sin\frac{\pi(1+n)}{l}t\right] \qquad (21)$$

will be a general solution. We differentiate (21) with respect to t.

$$u_t(x,t) = \sum_{n=0}^{\infty}\frac{\pi(1+n)}{l} sin\frac{\pi+\pi n}{l}x\left[b_n \cos\frac{\pi(1+n)}{l}t + a_n \sin\frac{\pi(1+n)}{l}t\right]$$

$u_t|_{t=0}$=0
according to the initial condition,

$$u_t(x,0) = \sum_{n=0}^{\infty}\frac{\pi(1+n)b_n}{l} sin\frac{\pi+\pi n}{l}x = 0 \quad \Rightarrow \quad b_n = 0$$

$u|_{t=0} = \sin 7x$ according to the initial condition and also taking into account that $\boldsymbol{l} = \pi$

$$u(x,0) = \sum_{n=0}^{\infty} a_n sin\frac{\pi+\pi n}{l}x = sin\,7x \;\Rightarrow\; u(x,0) = \sum_{n=0}^{\infty} a_n sin(1+n)\,x = sin\,7x$$

And from that a_6=1, a_n=0 bu yerda n=0,1,2,3,4,5,7,8………… n≠ 6.

From the obtained results, we write down the general solution of the equation as follows.

$$u(x,t) = sin\,7x\, cos\,7t$$

Conclusion. The general description of mixed and boundary problems for hyperbolic type equations was given. Information was given on the methods of solving the problems. A general solution of the boundary value problem put to the hyperbolic type equation was obtained.

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